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Symmetries, Currents and Conservation Laws of Self-Dual Gravity

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Abstract

We describe an infinite-dimensional algebra of hidden symmetries for the self-dual gravity equations. Besides the known diffeomorphism-type symmetries (affine extension of w_∞ algebra), this algebra contains new hidden symmetries, which are an affine extension of the Lorentz rotations. The full symmetry algebra has both Kac-Moody and Virasoro-like generators, whose exponentiation maps solutions of the field equations to other solutions. Relations to problems of string theories are briefly discussed.

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1. Introduction

The purpose of this paper is to describe a new infinite-dimensional algebra of hidden symmetries of the self-dual Einstein equations on a metric of signature $(++++)$ or $(++--)$. These equations define manifolds with self-dual Weyl tensor and vanishing Ricci tensor, which is equivalent to the self-duality equations for the Riemann tensor.

Four-dimensional self-dual Euclidean backgrounds often arise as the internal part of superstrings compactified to six dimensions in consideration of consistent string propagation (see, e.g., [1] and references therein). Self-dual gravity configurations also arise as consistent backgrounds for the $N=2$ closed string theory [2,3], and the $N=2$ string theory provides a quantization of the self-dual gravity model in a space-time with signature (2,2). Self-dual geometries are also important in compactifications of recently proposed 12-dimensional fundamental Y- and F-theories [4]. It is believed that discrete subgroups of the classical symmetry group of consistent string backgrounds are symmetries of string theory and that these subgroups of a large hidden symmetry group of string theory become visible for various compactifications [5]. Therefore hidden symmetries of the self-dual gravity equations are relevant to the symmetries of string theories.

The study of these symmetries is important for an understanding of non-perturbative properties and quantization of gravity and string theories. Euclidean solutions of the self-dual gravity equations (gravitational instantons) give a main contribution to a path integral of quantum gravity (see, e.g., [6]), and quantization of the self-dual gravity model itself may provide useful hints for full quantum gravity (see, e.g., [7]).

The self-duality equations on the curvature of a metric in four dimensions are an important example of a multidimensional integrable system, which can be solved by a twistor geometric construction [8–11]. The discussion of hidden symmetries of this model was started in the papers [12] on the basis of Plebański's equations [13], and has been continued by many authors (see, e.g., [14–18]). For the study of hidden symmetries the reformulation of the self-dual gravity equations as (reduced) self-dual Yang-Mills equations with infinite-dimensional gauge group was very useful [19,16,20] (see also the clear exposition in [21]). It was shown that the self-dual gravity equations are invariant with respect to a group, whose generators form the affine Lie algebra $w_\infty \otimes C[\lambda, \lambda^{-1}]$, $\lambda \in C$, associated with the Lie algebra w_∞ of area-preserving diffeomorphisms of a certain (null) surface [12,14–18].

We shall make a further step in the investigation of hidden symmetries of the self-dual gravity (SDG) equations. Our main results are the following:

- To each Lorentz rotation of the tangent space we associate an infinite number of new symmetries of the SDG equations and conserved currents. We show that these symmetries form a Kac-Moody-Virasoro type algebra, in fact the same as the one considered in [22]. These symmetries underlie the cancellation of almost all amplitudes in the theory of $N=2$ closed self-dual strings [2, 3].

- We define the action of the classical algebra w_∞ on the (conformal) tetrad and, using certain operator product expansion type formulae, we present a new derivation of the symmetry algebra $w_\infty \otimes C[\lambda] \subset w_\infty \otimes C[\lambda, \lambda^{-1}]$ of the SDG equations. We also describe the commutation relations between the generators of the 'old' and the 'new' symmetries.

- It is well-known that for metrics with 'rotational' Killing symmetry the SDG equations are reduced to the continual Toda equation ($sl(\infty)$-Toda field equation) [23], and for metrics with 'translational' Killing symmetry the SDG equations are reduced to the Gibbons-Hawking equations [24] in three dimensions. By reduction of the symmetry algebra of the SDG equations we obtain the well-known symmetry algebra w_∞ of the continual Toda equation [25] and the symmetry algebra of the Gibbons-Hawking equations, which has not appeared in the literature before.

- Recently, it was found that the T-duality transformation with respect to the rotational Killing vector fields (i.e. those which do not in general preserve the complex structure(s)) does not preserve the self-duality conditions, that leads to apparent violations of the $N = 4$ world-sheet supersymmetry [26] (see also [27]). The T-duality transformation with respect to the translational Killing vector fields (i.e. those which preserve the complex structure(s)) preserves the self-duality conditions. We show that the translational vector fields generate the Abelian loop group $LU(1) = C^\infty(S^1, U(1))$ of symmetries of the SDG equations, and the rotational vector fields generate the non-Abelian Virasoro symmetry group Diff(S^1). This "non-Abelian nature" of the rotational Killing vector fields underlies the nonpreservation of the local realizations of the world-sheet and space-time supersymmetries under the T-duality transformation with respect to such Killing vector fields.

In this paper we describe new hidden symmetries of the SDG equations omitting direct computations and writing out only the final formulae.

2. Manifest symmetries of self-dual gravity

Let M^4 be a complex four-dimensional manifold with a nondegenerate complex holomorphic metric g. We shall suppose that M^4 is oriented and denote by ω a complex holomorphic volume four-form. Consider the infinite-dimensional algebra sdiff(M^4) of volume-preserving vector fields on M^4. For $N = N^\mu\partial_\mu \in$ sdiff(M^4) $(\mu, \nu, ... = 1, ..., 4\ )$ the Lie derivative of ω along N should vanish (divergence free vector fields). Here, and throughout the paper, we use the Einstein summation convention.

Self-dual vacuum (i.e. Ricci flat) metrics may be constructed as follows [19]: For four pointwise linearly independent vector fields $B_\alpha \in$ sdiff(M^4) let us consider the following equations:

$$\frac{1}{2}\epsilon_{\alpha\beta}{}^{\gamma\delta}[B_\gamma, B_\delta] = [B_\alpha, B_\beta], \tag{1}$$

where $\alpha, \beta, ... = 1, ..., 4$ are Lorentz indices. If one introduces the vector fields

$$V_1 = \frac{1}{2}(B_1 - iB_2),\ V_{\tilde 1} = \frac{1}{2}(B_1 + iB_2),\ V_2 = \frac{1}{2}(B_3 - iB_4),\ V_{\tilde 2} = \frac{1}{2}(B_3 + iB_4), \tag{2}$$

then one may rewrite eqs.(1) in the form

$$[V_{\tilde 1}, V_{\tilde 2}] = 0,\ [V_{\tilde 1}, V_1] - [V_{\tilde 2}, V_2] = 0,\ [V_1, V_2] = 0. \tag{3}$$

Finally, let f be a scalar, a conformal factor, defined by $f^2 = \omega(V_1, V_2, V_{\tilde 1}, V_{\tilde 2})$. Then one may define a (contravariant) metric

$$g = f^{-2}(V_1 \otimes V_{\tilde{1}} + V_{\tilde{1}} \otimes V_1 - V_2 \otimes V_{\tilde{2}} - V_{\tilde{2}} \otimes V_2) \;\Leftrightarrow \tag{4a}$$

$$g^{\mu\nu} = f^{-2} g^{A\tilde{A}} (V_A^\mu V_{\tilde{A}}^\nu + V_A^\nu V_{\tilde{A}}^\mu), \tag{4b}$$

where $g^{1\tilde{1}} = g^{\tilde{1}1} = -g^{2\tilde{2}} = -g^{\tilde{2}2} = 1,\; A, B, ... = 1, 2,\; \tilde{A}, \tilde{B}, ... = 1, 2,$ and the Riemann tensor of this metric will be self-dual. Conversely, every self-dual vacuum metric arises in this way. For proofs and discussions see [19–21]. We call eqs.(3) (and eqs.(1)) the self-dual gravity (SDG) equations. Notice, that $\{f^{-1}V_{\tilde{A}}, f^{-1}V_A\}$ is a null tetrad for the self-dual vacuum metric (4).

An infinitesimal symmetry transformation of a system of partial differential equations is a map $\delta :\; s \to \delta s$, which to each solution s of the system assigns a solution δs of the linearized (around s) form of the system. The linearized form of the system may be derived by substituting $s + \epsilon\delta s$ into the system, and keeping only terms of the first order in the parameter ϵ. In particular, for eqs.(1) we obtain the following equations on δB_α:

$$\epsilon_{\alpha\beta}{}^{\gamma\sigma}[B_\gamma, \delta B_\sigma] = [B_\alpha, \delta B_\beta] + [\delta B_\alpha, B_\beta]. \tag{5}$$

For any two vector fields M, N in the algebra sdiff(M^4) we define the transformations of the vector fields $\{B_\alpha\}$ as follows:

$$\delta^0_M B_\alpha := [M, B_\alpha] \;\Rightarrow\; [\delta^0_M, \delta^0_N] B_\alpha = \delta^0_{[M,N]} B_\alpha. \tag{6}$$

Substituting (6) into (5) and using the Jacobi identities, it is not hard to show that $\delta^0_M B_\alpha$ satisfy eqs.(5), i.e. δ^0_M is a symmetry of eqs.(1).

Let us now consider global (not depending on coordinates) Lorentz rotations, which form the algebra $so(4, C) \simeq sl(2, C) \oplus sl(2, C)$, with the generators $\{W_i{}^\beta_\alpha\} = \{X_a{}^\beta_\alpha, X_{\hat{a}}{}^\beta_\alpha\}$:

$$[X_a, X_b] = f^c_{ab} X_c,\; [X_a, X_{\hat{b}}] = 0,\; [X_{\hat{a}}, X_{\hat{b}}] = f^{\hat{c}}_{\hat{a}\hat{b}} X_{\hat{c}}, \tag{7}$$

where $i, j, ... = 1, ..., 6;\; a, b, ... = 1, 2, 3;\; \hat{a}, \hat{b}, ... = 1, 2, 3;$ and $f^3_{12} = f^{\hat{3}}_{\hat{1}\hat{2}} = -f^1_{23} = -f^{\hat{1}}_{\hat{2}\hat{3}} = -f^2_{31} = -f^{\hat{2}}_{\hat{3}\hat{1}} = 1$ are the structure constants of the algebra $sl(2, C)$. Let us define the following transformations Δ_{W_i} of the vector fields $\{B_\alpha\}$:

$$\Delta_{W_i} B_\alpha := W_i{}^\beta_\alpha B_\beta \;\Rightarrow\; [\Delta_{W_i}, \Delta_{W_j}] B_\alpha = -\Delta_{[W_i, W_j]} B_\alpha. \tag{8}$$

One may consider $\{B_\alpha\}$ as a vector field with extra Lorentz index α. We write out the explicit formulae for the components $W_i{}^\beta_\alpha$ of the matrices W_i, defining the action of the transformations (8) on the vector field with components $\{V_{\tilde{A}}, V_A\}$ in the null frame:

$$\Delta_{X_1} V_{\tilde{1}} = -\frac{i}{2} V_{\tilde{2}},\; \Delta_{X_1} V_{\tilde{2}} = \frac{i}{2} V_{\tilde{1}},\; \Delta_{X_1} V_1 = \frac{i}{2} V_2,\; \Delta_{X_1} V_2 = -\frac{i}{2} V_1, \tag{9a}$$

$$\Delta_{X_2} V_{\tilde{1}} = \frac{1}{2} V_{\tilde{2}},\; \Delta_{X_2} V_{\tilde{2}} = \frac{1}{2} V_{\tilde{1}},\; \Delta_{X_2} V_1 = \frac{1}{2} V_2,\; \Delta_{X_2} V_2 = \frac{1}{2} V_1, \tag{9b}$$

$$\Delta_{X_3} V_{\tilde{1}} = -\frac{i}{2} V_{\tilde{1}},\; \Delta_{X_3} V_{\tilde{2}} = \frac{i}{2} V_{\tilde{1}},\; \Delta_{X_3} V_1 = \frac{i}{2} V_1,\; \Delta_{X_3} V_2 = -\frac{i}{2} V_2, \tag{9c}$$

$$\Delta_{X_{\hat{1}}} V_{\tilde{1}} = -\frac{i}{2} V_2,\ \Delta_{X_{\hat{1}}} V_{\tilde{2}} = -\frac{i}{2} V_1,\ \Delta_{X_{\hat{1}}} V_1 = \frac{i}{2} V_{\tilde{2}},\ \Delta_{X_{\hat{1}}} V_2 = \frac{i}{2} V_{\tilde{1}}, \tag{10a}$$

$$\Delta_{X_{\hat{2}}} V_{\tilde{1}} = \frac{1}{2} V_2,\ \Delta_{X_{\hat{2}}} V_{\tilde{2}} = \frac{1}{2} V_1,\ \Delta_{X_{\hat{2}}} V_1 = \frac{1}{2} V_{\tilde{2}},\ \Delta_{X_{\hat{2}}} V_2 = \frac{1}{2} V_{\tilde{1}}, \tag{10b}$$

$$\Delta_{X_{\hat{3}}} V_{\tilde{1}} = -\frac{i}{2} V_{\tilde{1}},\ \Delta_{X_{\hat{3}}} V_{\tilde{2}} = -\frac{i}{2} V_{\tilde{2}},\ \Delta_{X_{\hat{3}}} V_1 = \frac{i}{2} V_1,\ \Delta_{X_{\hat{3}}} V_2 = \frac{i}{2} V_2. \tag{10c}$$

It is obvious that

$$[\delta^0_M, \Delta_{W_i}] B_\alpha = 0, \tag{11}$$

i.e. the transformations (6) and (9), (10) commute.

The symmetries under the transformations (6) in the group SDiff(M^4) are gauge symmetries, and we may use them for the partial fixing of a coordinate system. Namely, from eqs.(3) it follows that one can always introduce coordinates $(y, z, \tilde{y}, \tilde{z})$ so that $V_{\tilde{1}}$ and $V_{\tilde{2}}$ become coordinate derivatives (Frobenius theorem), i.e. $V_{\tilde{A}} = \partial_{\tilde{A}}$, where $\partial_{\tilde{1}} \equiv \partial_{\tilde{y}},\ \partial_{\tilde{2}} \equiv \partial_{\tilde{z}}$. Then $[V_{\tilde{1}}, V_{\tilde{2}}] \equiv 0$, and the SDG eqs.(3) are reduced to

$$g^{\tilde{A}A} \partial_{\tilde{A}} V_A = 0 \ \Leftrightarrow\ \partial_{\tilde{1}} V_1 - \partial_{\tilde{2}} V_2 = 0, \tag{12a}$$

$$\epsilon^{AB} V_A V_B = 0 \ \Leftrightarrow\ [V_1, V_2] = 0, \tag{12b}$$

where $\epsilon^{12} = -\epsilon^{21} = 1$, and we have used the fact that $[\partial_{\tilde{A}}, K] = \partial_{\tilde{A}} K$ for any vector field K.

Remark. If we put $\tilde{y} = \bar{y},\ \tilde{z} = \bar{z}$ (where $\bar{y}$ and $\bar{z}$ are complex conjugate to y and z), then the solutions of eqs.(12) will define a tetrad on a real self-dual manifold with metric (4) of signature $(2, 2)$. If we put $\tilde{y} = \bar{y},\ \tilde{z} = -\bar{z}$, then solutions of eqs.(12) will define a tetrad on a real self-dual manifold with metric (4) of signature $(4, 0)$ (hyper-Kähler manifolds).

The vector fields $\{V_A\}$ from (12) may be parametrized by a scalar function (the only degree of freedom of self-dual metrics) in a different way, and then eqs.(12) will be reduced to different nonlinear equations on the scalar function (see [13–21]). For example, if we choose

$$V_1 = \Omega_{2\tilde{2}} \partial_1 - \Omega_{1\tilde{2}} \partial_2,\ V_2 = \Omega_{2\tilde{1}} \partial_1 - \Omega_{1\tilde{1}} \partial_2,\ \Omega_{A\tilde{A}} \equiv \partial_A \partial_{\tilde{A}} \Omega,\ \partial_1 \equiv \partial_y,\ \partial_2 \equiv \partial_z, \tag{13a}$$

then eqs.(12) are reduced to Plebański's first heavenly equation [13]:

$$\Omega_{1\tilde{2}} \Omega_{2\tilde{1}} - \Omega_{1\tilde{1}} \Omega_{2\tilde{2}} = 1. \tag{13b}$$

We shall not perform these reductions, because eqs.(12) are more fundamental than various scalar equations [13,16–18], obtained from (12) and carrying information about different parametrization of the vector fields $\{V_A\}$.

It is obvious that eqs.(12), derived from (3) by partial fixing of the coordinate system (in which $V_{\tilde{A}} = \partial_{\tilde{A}}$ should not change), will be not invariant under all the transformations from (6) and (8). The discussion of residual gauge invariance and of hidden symmetries will be the topic of the following Sections.

3. Affine extension of the $w_\infty \simeq$ sdiff(Σ^2) algebra

It is easy to see that the symmetries of eqs.(12) have to satisfy the equations:

$$\delta V_{\tilde{A}} \equiv \delta \partial_{\tilde{A}} = 0, \qquad (14a)$$

$$\partial_{\tilde{1}} \delta V_1 - \partial_{\tilde{2}} \delta V_2 = 0, \qquad (14b)$$

$$[V_1, \delta V_2] + [\delta V_1, V_2] = 0. \qquad (14c)$$

As to the transformations (6) from the algebra sdiff(M^4), it is evident that eqs.(12) will be invariant only under the subalgebra sdiff$(\Sigma^2) \subset$ sdiff(M^4) of those vector fields $M, N, ...$, which satisfy

$$\delta^0_M \partial_{\tilde{A}} := [\psi^0_M, \partial_{\tilde{A}}] = 0,\ \delta^0_M V_A := [\psi^0_M, V_A] \ \Rightarrow \qquad (15a)$$

$$[\delta^0_M, \delta^0_N]\partial_{\tilde{A}} = \delta^0_{[M,N]}\partial_{\tilde{A}} = 0,\ [\delta^0_M, \delta^0_N]V_A = \delta^0_{[M,N]}V_A. \qquad (15b)$$

Here we have denoted by Σ^2 the isotropic two-dimensional surfaces, parametrized by the coordinates $\{y, z\}$, and $\psi^0_M := M$.

It is not difficult to show that the transformations (15a) satisfy eqs.(14) and from eq.(14b) it follows that $\{\delta^0_M V_1, \delta^0_M V_2\}$ are two components of the conserved current $\delta^0_M V_A$. From (14b) it also follows that there exists a vector field ψ^1_M such that

$$\delta^0_M V_A \equiv [\psi^0_M, V_A] = \epsilon^{\tilde{B}}_A \partial_{\tilde{B}} \psi^1_M, \qquad (16)$$

where $\epsilon^{\tilde{B}}_A = g^{\tilde{B}B}\epsilon_{BA}$, $\epsilon_{12} = -\epsilon_{21} = 1 \Rightarrow \epsilon^{\tilde{2}}_1 = \epsilon^{\tilde{1}}_2 = 1$. Using ψ^1_M, we introduce the transformation δ^1_M by the formulae:

$$\delta^1_M \partial_{\tilde{A}} := 0,\ \delta^1_M V_A := [\psi^1_M, V_A]. \qquad (17)$$

It is not hard to verify that by virtue of eqs.(12), $\delta^1_M V_A$ satisfies eqs.(14). Therefore, $\delta^1_M V_A$ is also a conserved current. Now we may use a standard inductive procedure that was used, for example, for the construction of (nonlocal) currents of the chiral fields model [28,29]. Namely, let us suppose that we have constructed δ^n_M such that

$$\delta^n_M \partial_{\tilde{A}} := 0,\ \delta^n_M V_A := [\psi^n_M, V_A],\ n \geq 1. \qquad (18)$$

Assuming that the current $\delta^n_M V_A$ is conserved implies that there exists a vector field ψ^{n+1}_M such that

$$[\psi^n_M, V_A] = \epsilon^{\tilde{B}}_A \partial_{\tilde{B}} \psi^{n+1}_M. \qquad (19)$$

Using this we shall show that the $(n+1)$-th current $\delta^{n+1}_M V_A := [\psi^{n+1}_M, V_A]$ is conserved, which will complete the induction:

$$\partial_{\tilde{1}} \delta^{n+1}_M V_1 - \partial_{\tilde{2}} \delta^{n+1}_M V_2 = [\partial_{\tilde{1}} \psi^{n+1}_M, V_1] - [\partial_{\tilde{2}} \psi^{n+1}_M, V_2] =$$

$$= [[\psi^n_M, V_2], V_1] + [[V_1, \psi^n_M], V_2] = [[V_1, V_2], \psi^n_M] = 0, \qquad (20a)$$

$$[\delta^{n+1}_M V_1, V_2] + [V_1, \delta^{n+1}_M V_2] = [[\psi^{n+1}_M, V_1], V_2] + [V_1, [\psi^{n+1}_M, V_2]] = [\psi^{n+1}_M, [V_1, V_2]] = 0. \quad (20b)$$

Thus, for any $n \geq 1$ we construct a vector field ψ^n_M and a conserved current $\delta^n_M V_A$, starting from $\psi^0_M := M$ and $\delta^0_M V_A := [M, V_A]$.

Remark. Using (4), (15), (18) and (19), one may show by direct calculations that $\delta^0_M g^{\mu\nu} = \mathcal{L}_{\psi^0_M} g^{\mu\nu}$, but for $n \geq 1$ $\delta^n_M g^{\mu\nu} \neq \mathcal{L}_{\psi^n_M} g^{\mu\nu}$, where $\mathcal{L}_{\psi^n_M}$ is a Lie derivative

along the vector field ψ^n_M. This means that δ^0_M is a gauge symmetry (an infinitesimal diffeomorphism), and δ^n_M with $n \geq 1$ is not a gauge symmetry.

Having an infinite number of vector fields ψ^n_M on M^4, one can introduce the vector field $\psi_M(y,z,\tilde{y},\tilde{z},\lambda) := \sum_{n=0}^{\infty}\lambda^n\psi^n_M(y,z,\tilde{y},\tilde{z})$, depending on the complex parameter $\lambda \in C$. Then the infinite number of equations (19) (*recurrence relations*) may be rewritten as two linear equations on $\psi_M(\lambda)$:

$$\begin{array}{l}\partial_{\tilde{1}}\psi_M + \lambda[V_2,\psi_M] = 0\\ \partial_{\tilde{2}}\psi_M + \lambda[V_1,\psi_M] = 0\end{array} \iff \partial_{\tilde{A}}\psi_M + \lambda\epsilon^B_{\tilde{A}}[V_B,\psi_M] = 0, \tag{21}$$

where $\epsilon^A_{\tilde{B}} = g^{A\tilde{A}}\epsilon_{\tilde{A}\tilde{B}},\ \epsilon_{\tilde{1}\tilde{2}} = -\epsilon_{\tilde{2}\tilde{1}} = 1 \ \Rightarrow\ \epsilon^2_{\tilde{1}} = \epsilon^1_{\tilde{2}} = 1.$

Remark. Equations (21) can be considered as a linear system (Lax pair) for the SDG equations (12), because eqs.(12) are the compatibility conditions of eqs.(21). As a 'canonical' vector field one may choose, e.g., ∂_A and consider the linear equations (21) on ψ_{∂_A}.

Instead of an infinite number of symmetry generators δ^n_M one may introduce the generator $\delta_M(\lambda) := \sum_{n=0}^{\infty}\lambda^n\delta^n_M$, depending on a complex 'spectral' parameter $\lambda \in C$. It is evident that $\delta^n_M = (2\pi i)^{-1}\oint_{C'} d\lambda\ \lambda^{-n-1}\delta_M(\lambda)$, where C' is a contour in the λ-plane about the origin. Using $\psi_M(\lambda)$ and $\delta_M(\lambda)$, formulae (15a) and (18) may be rewritten in the form of a one-parameter family of infinitesimal transformations

$$\delta_M(\lambda)V_{\tilde{A}} := 0,\ \delta_M(\lambda)V_A := [\psi_M(\lambda), V_A], \tag{22}$$

which are symmetries of eqs.(12) for each $M \in \mathrm{sdiff}(\Sigma^2)$.

Now we are interested in the algebraic properties of the symmetries (22). It is not difficult to show that

$$\begin{aligned}\delta_M(\lambda)\delta_N(\zeta)V_A &:= \delta_M(\lambda)(V_A + \delta_N(\zeta)V_A) - \delta_M(\lambda)V_A = \\ &= [\psi_M(\lambda) + \delta_N(\zeta)\psi_M(\lambda), V_A + \delta_N(\zeta)V_A] - \delta_M(\lambda)V_A \\ &\cong [\delta_N(\zeta)\psi_M(\lambda), V_A] + [\psi_M(\lambda), \delta_N(\zeta)V_A],\end{aligned} \tag{23a}$$

$$\delta_N(\zeta)\delta_M(\lambda)V_A = [\delta_M(\lambda)\psi_N(\zeta), V_A] + [\psi_N(\zeta), \delta_M(\lambda)V_A]. \tag{23b}$$

Then the commutator of two symmetries is equal to

$$[\delta_M(\lambda),\delta_N(\zeta)]V_A = [\delta_N(\zeta)\psi_M(\lambda) - \delta_M(\lambda)\psi_N(\zeta) + [\psi_M(\lambda),\psi_N(\zeta)], V_A]. \tag{24}$$

Accordingly, for the variation $\delta_N(\zeta)\psi_M(\lambda)$ we have the following equations

$$\partial_{\tilde{A}}\delta_N(\zeta)\psi_M(\lambda) + \lambda\epsilon^B_{\tilde{A}}[V_B, \delta_N(\zeta)\psi_M(\lambda)] = \lambda\epsilon^B_{\tilde{A}}[\psi_M(\lambda), \delta_N(\zeta)V_B], \tag{25}$$

the solutions of which have the form (cf. [29]):

$$\delta_N(\zeta)\psi_M(\lambda) = \frac{\zeta}{\lambda-\zeta}(\psi_{[M,N]}(\zeta) - [\psi_M(\lambda),\psi_N(\zeta)]) \ \Rightarrow$$

$$\delta_M(\lambda)\psi_N(\zeta) = \frac{\lambda}{\lambda-\zeta}(\psi_{[M,N]}(\lambda) - [\psi_M(\lambda),\psi_N(\zeta)]). \tag{26}$$

Substituting (26) into (24), we obtain

$$[\delta_M(\lambda), \delta_N(\zeta)] = \frac{1}{\lambda - \zeta}(\lambda \delta_{[M,N]}(\lambda) - \zeta \delta_{[M,N]}(\zeta)) \;\Rightarrow\; [\delta^m_M, \delta^n_N] = \delta^{m+n}_{[M,N]},\;\; m,n \geq 0, \quad (27)$$

when we consider the action on V_A and $V_{\tilde{A}}$. The algebra (27) is the affine extension sdiff$(\Sigma^2)\otimes C[\lambda]$ of the algebra sdiff(Σ^2) of area-preserving diffeomorphisms. Formulae (27) give us commutators between half of the generators of the affine Lie algebra sdiff$(\Sigma^2)\otimes C[\lambda, \lambda^{-1}]$.

Remarks.

1. The described algebra sdiff$(\Sigma^2)\otimes C[\lambda]$ of symmetries of the SDG equations (12) is known. But the formulae (15)–(27), describing the action of these symmetries on the (conformal) tetrad $\{V_{\tilde{A}}, V_A\}$, are new. These formulae may be useful for applications.

2. In the described algebra there is an Abelian subalgebra with generators $\{\delta^n_{\partial_A}\}$, where $\{\partial_A\} = \{\partial_y, \partial_z\}$. In the usual way (see, e.g., [30]), one can associate to this algebra of Abelian symmetries the hierarchy of the SDG equations (cf. [15,17] for other approaches).

3. We restrict our attention to the subalgebra sdiff$(\Sigma^2)\otimes C[\lambda]$ of the symmetry algebra sdiff$(\Sigma^2)\otimes C[\lambda, \lambda^{-1}]$. The rest will be obtained if we choose the coordinates in such a way that V_A will be coordinate derivatives (in Sec.2 and Sec.3 $V_{\tilde{A}}$ were the coordinate derivatives) and consider symmetries of eqs.(3) after this 'dual' partial fixing of coordinates.

4. Hidden symmetries from Lorentz rotations

The explicit form of the infinitesimal transformations of the vector fields $\{V_{\tilde{A}}, V_A\}$ under the action of the Lorentz group $SO(4,C)$ was written out in (9) and (10). From (9), (10) one can see that $\Delta_{W_i}\partial_{\tilde{A}} \neq 0$, i.e. these transformations, being the symmetry of eqs.(3), are not the symmetry of eqs.(12). In other words, the transformations (9) and (10) do not preserve the chosen gauge. Nevertheless the Lorentz symmetry can be restored by compensating transformations from the diffeomorphism group SDiff(M^4). The point is that from formulae (9), (10) it follows that

$$\partial_{\tilde{1}}(\Delta_{W_i}\partial_{\tilde{2}}) - \partial_{\tilde{2}}(\Delta_{W_i}\partial_{\tilde{1}}) = 0, \quad (28)$$

for any $W_i \in so(4,C)$. This means that there exist vector fields $\{\psi^0_{W_i}\} = \{\psi^0_{X_a}, \psi^0_{X_{\hat{a}}}\}$ such that

$$\Delta_{W_i}\partial_{\tilde{1}} = \partial_{\tilde{1}}\psi^0_{W_i},\;\; \Delta_{W_i}\partial_{\tilde{2}} = \partial_{\tilde{2}}\psi^0_{W_i}, \quad (29)$$

and one can define the transformation

$$\delta^0_{W_i}\partial_{\tilde{A}} := \Delta_{W_i}\partial_{\tilde{A}} + [\psi^0_{W_i}, \partial_{\tilde{A}}] = 0, \quad (30a)$$

$$\delta^0_{W_i}V_A := \Delta_{W_i}V_A + [\psi^0_{W_i}, V_A], \quad (30b)$$

satisfying the following commutator relations

$$[\delta^0_{W_i},\delta^0_{W_j}]V_{\tilde A}=\delta^0_{[W_i,W_j]}V_{\tilde A}=0,\ [\delta^0_{W_i},\delta^0_{W_j}]V_A=\delta^0_{[W_i,W_j]}V_A. \tag{31}$$

Notice that the equality to zero in (30a) follows from the definition (29) of the vector fields $\{\psi^0_{W_i}\}$.

Remark. Using eqs. (4) and (30), one can show by direct computation that $\delta^0_{W_i}$ acts on $g^{\mu\nu}$ as a Lie derivative, i.e. $\delta^0_{W_i}g^{\mu\nu}=\mathcal{L}_{\psi^0_{W_i}}g^{\mu\nu}$. Therefore, if one defines the action of $\delta^0_{W_i}$ on the vector fields ψ_M from the linear system (21), then one can develop the method of reduction for the SDG equations (12) and the linear system (21) for them, analogous to the method developed for the self-dual Yang-Mills model [31,32].

It is not difficult to show that (30) satisfy eqs.(14). So, $\delta^0_{W_i}V_A$ is a conserved current and from (14b) it follows that there exists a vector field $\psi^1_{W_i}$ such that

$$\delta^0_{W_i}V_A\equiv\Delta_{W_i}V_A+[\psi^0_{W_i},V_A]=\epsilon_A^{\tilde B}\partial_{\tilde B}\psi^1_{W_i}. \tag{32}$$

Let us define in full analogy with Sec.3 the transformations

$$\delta^1_{W_i}\partial_{\tilde A}:=0,\ \delta^1_{W_i}V_A:=[\psi^1_{W_i},V_A]. \tag{33}$$

One can verify that (33) is a symmetry of eqs.(12). Now with the help of the inductive procedure, identical to the one described in Sec.3, it is not difficult to show that the transformations

$$\delta^{n+1}_{W_i}\partial_{\tilde A}:=0,\quad \delta^{n+1}_{W_i}V_A:=[\psi^{n+1}_{W_i},V_A] \tag{34}$$

are symmetries of eqs.(12), if

$$\delta^n_{W_i}V_A\equiv[\psi^n_{W_i},V_A]=\epsilon_A^{\tilde B}\partial_{\tilde B}\psi^{n+1}_{W_i},\ n\ge 1, \tag{35}$$

is a conserved current.

One may introduce the generating vector field $\psi_{W_i}(y,z,\tilde y,\tilde z,\zeta):=\sum_{n=0}^{\infty}\zeta^n\psi^n_{W_i}(y,z,\tilde y,\tilde z)$, $\zeta\in C$. Then the recurrence relations (35) can be collected into the following two linear equations

$$[\partial_{\tilde A}+\zeta\epsilon_{\tilde A}^{B}V_B,\psi_{W_i}(\zeta)]=\Delta_{W_i}\partial_{\tilde A}+\zeta\epsilon_{\tilde A}^{B}\Delta_{W_i}V_B. \tag{36}$$

Analogously, introducing $\delta_{W_i}(\zeta):=\sum_{n=0}^{\infty}\zeta^n\delta^n_{W_i}$, we obtain a one-parameter family of infinitesimal transformations

$$\delta_{W_i}(\zeta)\partial_{\tilde A}:=0,\ \delta_{W_i}(\zeta)V_A:=[\psi_{W_i}(\zeta),V_A]+\Delta_{W_i}V_A. \tag{37}$$

For each $W_i\in so(4,C)$ these transformations are new 'hidden symmetries' of the SDG equations (12).

After some calculations we have the following expression for the commutator of two symmetries

$$[\delta_{W_i}(\lambda),\delta_{W_j}(\zeta)]V_A=\epsilon_A^{\tilde B}\partial_{\tilde B}\{\frac{1}{\lambda}\delta_{W_j}(\zeta)\psi_{W_i}(\lambda)-\frac{1}{\zeta}\delta_{W_i}(\lambda)\psi_{W_j}(\zeta)\}+$$

$$+\frac{1}{\zeta}\epsilon_A^{\tilde B}\Delta_{W_j}{}^{C}_{\tilde B}\delta_{W_i}(\lambda)V_C-\frac{1}{\lambda}\epsilon_A^{\tilde B}\Delta_{W_i}{}^{C}_{\tilde B}\delta_{W_j}(\zeta)V_C. \tag{38}$$

From eqs.(36) one obtains the equations for the variation $\delta_{W_i}(\lambda)\psi_{W_j}(\zeta)$:

$$[\partial_{\tilde{A}}+\zeta\epsilon_{\tilde{A}}^{B}V_B,\delta_{W_i}(\lambda)\psi_{W_j}(\zeta)]=\zeta\epsilon_{\tilde{A}}^{B}[\psi_{W_j}(\zeta),\delta_{W_i}(\lambda)V_B]+(\Delta_{W_j}{}^{B}_{\tilde{A}}+\zeta\epsilon_{\tilde{A}}^{C}\Delta_{W_j}{}^{B}_{C})\delta_{W_i}(\lambda)V_B. \quad (39)$$

Using the identities

$$\Delta_{X_a}{}^{B}_{\tilde{A}}=0, \quad \Delta_{X_{\hat{a}}}{}^{B}_{\tilde{A}}+\zeta\epsilon_{\tilde{A}}^{C}\Delta_{X_{\hat{a}}}{}^{B}_{C}=(Z_{\hat{a}}^{\zeta}-\frac{\zeta}{2}\dot{Z}_{\hat{a}}^{\zeta})\epsilon_{\tilde{A}}^{B}, \quad (40)$$

where $Z_{\hat{a}}^{\zeta}$ are the components of vector fields

$$Z_{\hat{a}}=Z_{\hat{a}}^{\zeta}\partial_{\zeta},\ [Z_{\hat{a}},Z_{\hat{b}}]=f_{\hat{a}\hat{b}}^{\hat{c}}Z_{\hat{c}}$$

$$Z_{\hat{1}}^{\zeta}=-\frac{i}{2}(1+\zeta^2),\ Z_{\hat{2}}^{\zeta}=\frac{1}{2}(1-\zeta^2),\ Z_{\hat{3}}^{\zeta}=i\zeta,\ \dot{Z}_{\hat{a}}^{\zeta}\equiv\frac{d}{d\zeta}Z_{\hat{a}}^{\zeta}, \quad (41)$$

we obtain the solution of eqs. (39) in the form

$$\delta_{W_i}(\lambda)\psi_{W_j}(\zeta)=\frac{\zeta}{(\lambda-\zeta)}\{\psi_{[W_i,W_j]}(\zeta)-[\psi_{W_i}(\lambda),\psi_{W_j}(\zeta)]-W_i^{\zeta}\partial_{\zeta}\psi_{W_j}(\zeta)\}+$$

$$+\frac{\lambda}{(\lambda-\zeta)^2}W_j^{\zeta}\{\psi_{W_i}(\lambda)-\psi_{W_i}(\zeta)\}, \quad (42)$$

where $W_a^{\zeta}:=0,\ W_{\hat{a}}^{\zeta}:=Z_{\hat{a}}^{\zeta}$.

Substituting (42) into (38), we obtain the following expression for the commutator of two successive infinitesimal transformations:

$$[\delta_{W_i}(\lambda),\delta_{W_j}(\zeta)]V_A=\frac{1}{(\lambda-\zeta)}\{\lambda\delta_{[W_i,W_j]}(\lambda)-\zeta\delta_{[W_i,W_j]}(\zeta)\}V_A+$$

$$+\frac{1}{(\lambda-\zeta)^2}\{\frac{\zeta}{\lambda}W_i^{\lambda}(\zeta\delta_{W_j}(\zeta)-\lambda\delta_{W_j}(\lambda))+\frac{\lambda}{\zeta}W_j^{\zeta}(\zeta\delta_{W_i}(\zeta)-\lambda\delta_{W_i}(\lambda))\}V_A+$$

$$+\frac{1}{\zeta}\epsilon_{A}^{\tilde{B}}\Delta_{W_j}{}^{C}_{\tilde{B}}\delta_{W_i}(\lambda)V_C-\frac{1}{\lambda}\epsilon_{A}^{\tilde{B}}\Delta_{W_i}{}^{C}_{\tilde{B}}\delta_{W_j}(\zeta)V_C+$$

$$+\frac{1}{(\lambda-\zeta)}\{W_i^{\zeta}\partial_{\zeta}(\zeta\delta_{W_j}(\zeta))+W_j{}^{\lambda}\partial_{\lambda}(\lambda\delta_{W_i}(\lambda))\}V_A. \quad (43)$$

In order to rewrite (43) in terms of the generators $\delta_{W_i}^{n}=(2\pi i)^{-1}\oint_{C'}d\lambda\ \lambda^{-n-1}\delta_{W_i}(\lambda)$, it is convenient to introduce $Y_0,Y_{\pm}$ instead of $X_{\hat{a}}$:

$$Y_0:=iX_{\hat{3}},\ Y_+:=-iX_{\hat{1}}+X_{\hat{2}},\ Y_-:=-iX_{\hat{1}}-X_{\hat{2}},\ [Y_{\pm},Y_0]=\pm Y_{\pm},\ [Y_+,Y_-]=2Y_0. \quad (44)$$

Using (43) and (44), we obtain

$$[\delta_{X_a}^{m},\delta_{X_b}^{n}]=\delta_{[X_a,X_b]}^{m+n},\ m,n,...\geq 0, \quad (45)$$

$$[\delta_{Y_0}^{m},\delta_{Y_0}^{n}]=2(m-n)\delta_{Y_0}^{m+n},\ [\delta_{Y_+}^{m},\delta_{Y_+}^{n}]=2(m-n)\delta_{Y_+}^{m+n-1},\ [\delta_{Y_-}^{m},\delta_{Y_-}^{n}]=2(m-n)\delta_{Y_-}^{m+n+1},$$

$$[\delta_{Y_0}^{m},\delta_{Y_+}^{n}]=\delta_{[Y_0,Y_+]}^{m+n}+2m\delta_{Y_0}^{m+n-1}-2n\delta_{Y_+}^{m+n},$$

$$[\delta_{Y_0}^{m},\delta_{Y_-}^{n}]=\delta_{[Y_0,Y_-]}^{m+n}+2m\delta_{Y_0}^{m+n+1}-2n\delta_{Y_-}^{m+n},$$

$$[\delta^m_{Y_+},\delta^n_{Y_-}]=\delta^{m+n}_{[Y_+,Y_-]}+2m\delta^{m+n+1}_{Y_+}-2n\delta^{m+n-1}_{Y_-}, \tag{46}$$

$$[\delta^m_{Y_0},\delta^n_{X_a}]=-2n\delta^{m+n}_{X_a},\ [\delta^m_{Y_+},\delta^n_{X_a}]=-2n\delta^{m+n-1}_{X_a},\ [\delta^m_{Y_-},\delta^n_{X_a}]=-2n\delta^{m+n+1}_{X_a}, \tag{47}$$

Formulae (45) mean that $\{\delta^m_{X_a}\}$ are the generators of the affine Lie algebra $sl(2,C)\otimes C[\lambda]$, which is the subalgebra in $sl(2,C)\otimes C[\lambda,\lambda^{-1}]$. From (46) one can see that $\delta^m_{Y_0},\delta^m_{Y_+}$ and $\delta^m_{Y_-}$ generate three different Virasoro-like subalgebras of the symmetry algebra.

Thus, the new algebra of 'hidden symmetries' of the SDG equations with generators $\{\delta^m_{X_1},\delta^m_{X_2},\delta^m_{X_3},\delta^m_{Y_0},\delta^m_{Y_+},\delta^m_{Y_-}\}$ forms a Kac-Moody-Virasoro algebra with commutation relations (45) – (47). This algebra has the same commutation relations as a subalgebra of the symmetry algebra of the self-dual Yang-Mills equations [22].

5. Commutators of symmetries and comments

In Sec.4 and Sec.3 eqs.(36) on $\psi_{W_j},\ W_j\in so(4,C)$, and eqs.(21) on ψ_M, $M\in \mathrm{sdiff}(\Sigma^2)$, have been written out. From these equations one can derive the equations for the variations of the vector fields ψ_{W_j} and ψ_M:

$$[\partial_{\tilde A}+\zeta\epsilon^B_{\tilde A}V_B,\delta_M(\lambda)\psi_{W_j}(\zeta)]=\zeta\epsilon^B_{\tilde A}[\psi_{W_j}(\zeta),\delta_M(\lambda)V_B]+$$

$$+(\Delta_{W_j}{}^B_{\tilde A}+\zeta\epsilon^C_{\tilde A}\Delta_{W_j}{}^B_C)\delta_M(\lambda)V_B, \tag{48a}$$

$$[\partial_{\tilde A}+\lambda\epsilon^B_{\tilde A}V_B,\delta_{W_j}(\zeta)\psi_M(\lambda)]=\lambda\epsilon^B_{\tilde A}[\psi_M(\lambda),\delta_{W_j}(\zeta)V_B]. \tag{48b}$$

We have the following solutions of these equations:

$$\delta_M(\lambda)\psi_{W_j}(\zeta)=\frac{\zeta}{\lambda-\zeta}[\psi_{W_j}(\zeta),\psi_M(\lambda)]+\frac{\lambda}{(\lambda-\zeta)^2}W^\zeta_j\{\psi_M(\lambda)-\psi_M(\zeta)\}, \tag{49a}$$

$$\delta_{W_j}(\zeta)\psi_M(\lambda)=\frac{\lambda}{\lambda-\zeta}\{[\psi_{W_j}(\zeta),\psi_M(\lambda)]+W^\lambda_j\partial_\lambda\psi_M(\lambda)\}. \tag{49b}$$

Then after some computation we obtain the following expression for the commutator

$$[\delta_M(\lambda),\delta_{W_j}(\zeta)]V_A=\frac{1}{(\lambda-\zeta)^2}\{\frac{\lambda}{\zeta}W^\zeta_j(\zeta\delta_M(\zeta)-\lambda\delta_M(\lambda))\}V_A+$$

$$+\frac{1}{\zeta}\epsilon^{\tilde B}_A\Delta_{W_j}{}^C_{\tilde B}\delta_M(\lambda)V_C+\frac{1}{(\lambda-\zeta)}W^\lambda_j\partial_\lambda(\lambda\delta_M(\lambda))V_A. \tag{50}$$

Using the definition of $\delta^m_M,\ \delta^n_{W_j}$, formulae (44) and the commutator (50), we obtain

$$[\delta^m_{X_a},\delta^n_M]=0,\ [\delta^m_{Y_0},\delta^n_M]=-2n\delta^{m+n}_M, \tag{51a}$$

$$[\delta^m_{Y_+},\delta^n_M]=-2n\delta^{m+n-1}_M,\ [\delta^m_{Y_-},\delta^n_M]=-2n\delta^{m+n+1}_M,\ m,n,...\geq 0. \tag{51b}$$

Thus, the 'hidden symmetries' of the SDG equations (12) form the infinite-dimensional Lie algebra with the commutation relations (27), (45)–(47) and (51).

Notice, that taking $\zeta=0$ in (49b), we obtain the action of $\delta^0_{W_j}$ on $\psi_M(\lambda)$:

$$\delta^0_{X_a}\psi_M(\lambda)=[\psi^0_{X_a},\psi_M(\lambda)],\ \delta^0_{X_{\hat a}}\psi_M(\lambda)=[\psi^0_{X_{\hat a}},\psi_M(\lambda)]+Z^\lambda_{\hat a}\partial_\lambda\psi_M(\lambda). \tag{52}$$

From (52) it follows that $\delta^0_{X_a}$ acts on $\psi_M(\lambda)$ as the Lie derivative along the vector field $\psi^0_{X_a}$, and $\delta^0_{X_{\hat{a}}}$ acts on $\psi_M(\lambda)$ as the Lie derivative along the "lifted" vector field $\psi^0_{X_{\hat{a}}} + Z_{\hat{a}}$ (cf. [31,32,22] for the SDYM case). The reason is that the vector fields $\psi^0_{X_a}$, defined on the manifold M^4, have the trivial lift on the twistor space $M^4 \times B^2$ ($B^2 \simeq S^2$ for Euclidean signature and $B^2 \simeq H^2$ for the signature (2, 2)), and the lift of the vector fields $\psi^0_{X_{\hat{a}}}$ is nontrivial.

Remember that $\delta^0_{W_j}$ act on the metric as Lie derivatives: $\delta^0_{W_j} g^{\mu\nu} = \mathcal{L}_{\psi^0_{W_j}} g^{\mu\nu}$. Therefore one can consider reductions of the SDG equations (12) and of the linear system (21) for them by imposing the invariance conditions of the tetrad and of ψ_M with respect to the vector fields $\{\psi^0_{W_j}\}$. For example, conditions $\delta^0_{Y_0} V_A = \mathcal{L}_{\psi^0_{Y_0}} V_A = 0$ reduce the SDG equations to the $sl(\infty)$-Toda field equation (see, e.g., [23,26]). Since $\delta^0_{Y_0}$ generates the Lie algebra $\text{diff}(S^1) = \{\delta^n_{Y_0}\}$ of the group $\text{Diff}(S^1)$, then the space of solutions of the $sl(\infty)$-Toda field equation can be obtained from the space $\mathcal{M}$ of solutions of the SDG equations by factorization under the group $\text{Diff}(S^1)$. The imposing of $\delta^0_{Y_0}$-symmetry (from which there also follow the symmetries under $\delta^n_{Y_0},\ n \geq 1$), automatically reduces the algebra of hidden symmetries of the SDG equations to the well-known algebra $w_\infty \simeq \text{sdiff}(\Sigma^2)$ of symmetries of the $sl(\infty)$-Toda field equation [25]. Namely, only the subalgebra with generators $\{\delta^0_M\}$ will preserve the symmetry condition (this algebra is a normalizer of the algebra $\text{diff}(S^1)$ in the symmetry algebra).

Analogously, the conditions $\delta^0_{X_3} V_A = \mathcal{L}_{\psi^0_{X_3}} V_A = 0$ reduce the SDG equations to the Gibbons-Hawking equations [24,26], describing, in particular, ALE gravitational instantons. From the symmetry with respect to $\delta^0_{X_3}$ there follows the symmetry with respect to the whole algebra $\{\delta^n_{X_3}\}$ of the Abelian loop group $LU(1) = C^\infty(S^1, U(1))$. Therefore, the space of solutions of the Gibbons-Hawking equations is obtained from the space $\mathcal{M}$ of solutions of the SDG equations by factorization under the group $LU(1)$. From the commutation relations (45)–(47) and (51) it follows that the subalgebra with generators $\{\delta^n_M, \delta^n_{Y_0}, \delta^n_{Y_+}, \delta^n_{Y_-},\ n \geq 0\}$ will preserve the symmetry condition. This algebra has not been described in the literature before.

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